Differential Equations
Quiz 3
Solutions and Modeling

MTH 221
April 14, 2009
Due: Monday April 20, 2009
Max Pts: 50

Work on these problems and have them ready to turn in by Wednesday April 23, 2008. I expect a well organized and easy to read solution set. You are more than welcome to ask me question and/or talk with one another. Happy problem solving and "May the force be with you".

1. Solve the IVP

\[ y' = \frac{2 - e^x}{3 + 2y} \quad y(0) = 0 \]

and determine where the solution attains its maximum. Make an appropriate sketch to validate these findings.

2. Suppose you are having a dinner party for a large group of people, and you decide to make 2 gallons of chili. The recipe calls for 2 teaspoons of hot sauce per gallon, but you misread the instructions and put in 2 tablespoons of hot sauce per gallon. (Since each tablespoon is 3 teaspoons, you have put in 6 teaspoons per gallon, which is a total of 12 teaspoons of hot sauce in the chili.) You don’t want to throw the chili out because there isn’t much else to eat (and some people like hot chili), so you serve the chili anyway. However, as each person takes some chili, you fill up the pot with beans and tomatoes without hot sauce until the concentration of hot sauce agrees with the recipe. Suppose guests take 1 cup of chili per minute from the pot (there are 16 cups in a gallon), how long will it take to get the chili back to the recipe’s concentration of hot sauce? How many cups of chili will have been taken from the pot?

3. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see figure below): a resistive force \( R \), a buoyant force \( B \), and its weight \( w \) due to gravity. The buoyant force is equal to the weight of the fluid displayed by the object. For a slowly moving spherical body of radius \( a \), the resistive force is given by Stokes Law, \( R = 6\pi\mu a|v| \), where \( v \) is the velocity of the body, and \( \mu \) is the coefficient of viscosity of the surrounding fluid.

(a) Set-up a differential equation that explains the variation of the speed of the sphere \( v \) with time. First, determine the total force \( F \) acting on the sphere. Then use Newton’s second law to write \( F = m \frac{dv}{dt} \), where \( m \) is the mass of the sphere. Take the downward direction to be positive.

(b) Determine the limiting velocity of the sphere of radius \( a \) and density \( \rho \) falling freely in a medium of density \( \rho' \) and coefficient of viscosity \( \mu \).
4. Boyce and Diprima section 2.3 problem 32, the classic Brachistochrone problem.

One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point \( P \) to another \( Q \), the second point being lower than the first but not directly beneath it (see figure). This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Liebniz, and the Marquis de L’Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem it is convenient to take the origin as the upper point \( P \) and to orient the axes shown in the figure. The lower point \( Q \) has coordinates \((x_0, y_0)\). It is then possible to show that the curve of minimum time is given by a function \( y = \phi(x) \) that satisfies the differential equation

\[(1 + y'^2)y = k^2 \tag{1}\]

where \( k^2 \) is a certain positive constant to be determined later.

(a) Solve Eqn. (1) for \( y' \). Why is it necessary to choose the positive square root?

(b) Introduce the new variable \( t \) by the relation

\[y = k^2 \sin^2 t \tag{2}\]

Show the equation in found in part (a) then takes the form

\[2k^2 \sin^2 t \ dt = dx \tag{3}\]

(c) Letting \( \theta = 2t \), show that the solution to 3 for which \( x = 0 \) when \( y = 0 \) is given by

\[x = k^2(\theta - \sin(\theta))/2, \quad y = k^2(1 - \cos(\theta))/2, \tag{4}\]

Equations (4) are parametric equations of the solution of Eqn. (1) that passes through \((0,0)\). The graph of Eqn. (4) is called a \textbf{cycloid}.

(d) If we make a proper choice of the constant \( k \), then the cycloid also passes through the point \((x_0, y_0)\) and is the solution of the brachistochrone problem. Find \( k \) if \( x_0 = 1 \) and \( y_0 = 2 \).